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A New Property of Critical Imperfect Graphs and some Consequences

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We prove a new property of critical imperfect graphs. As a consequence, we define a new class of perfect graphs. This class contains perfectly orderable graphs and graphs in which that every odd cycle has two chords.

1. INTRODUCTION

A graph G is said to be γ-perfect if the chromatic number $\gamma(G')$ of any induced subgraph G' of G is equal to $\omega(G')$, the size of the largest complete subgraph in G'. G is said to be θ-perfect if and only if its complementary $\bar{G}$ is γ-perfect.

Lovasz [12] or [13] proved that a graph is γ-perfect if and only if it is θ-perfect. We can speak of perfect graphs instead of θ or γ-perfect graphs.

The well-known 'strong perfect graph conjecture' due to C. Berge (see for instance [2]), states that G is perfect if and only if G contains neither an odd hole (i.e. an odd cycle without a chord), nor an antihole (i.e. a hole in $\bar{G}$). For further information on the strong perfect graph conjecture the reader is referred to [2].

In the present paper we give a new property of critical imperfect graphs and as a consequence, we define a new class of perfect graphs. this new class contains two known classes of perfect graphs: perfectly orderable graphs defined by V. Chvatal [6] and graphs such that every odd cycle has two chords [14].

2. DEFINITIONS AND NOTATIONS

Definitions and notations are classical, see [1]. We consider here only finite simple undirected graphs unless otherwise specified. An *induced path* between two vertices x and y is a path $(x_0 = x, x_1, \ldots, x_m = y)$ such that (x_i, x_j), $i < j$, is an edge of G if and only if $j = i + 1$, $i = 0, 1, \ldots, m - 1$. An induced path is *even* if it contains an even number of edges. The *length* of a path is its number of edges.

A graph G is said to be *critical imperfect* if any proper induced subgraph of G is perfect and if G itself is not perfect.

The size of the complete largest subgraph in G is denoted by $\omega(G)$, the size of the largest independent subset by $\alpha(G)$, the chromatic number of G by $\gamma(G)$. The set of vertices of G adjacent to a vertex x is denoted by $\Gamma_G(x)$. If G is directed, we denote by $\Gamma_G^+(x)$, (resp. $\Gamma_G^-(x)$), the set of vertices y such that (x, y), (resp. (y, x)), is an arc of G.

3. THE RESULTS

We shall prove that a critical imperfect graph is such that each pair of different vertices is joined by an odd induced path. In order to prove this result, we need two lemmas.

LEMMA 1. *Let G be a graph. If two non adjacent vertices x and y of G are not linked by an induced path of length* 3, *the graph G' obtained from G by identifying x and y satisfies $\omega(G') = \omega(G)$.*

PROOF. Clearly $\omega(G') \geqslant \omega(G)$. Suppose we have $\omega(G') > \omega(G)$, then there must exist in G some complete subgraph on a set of vertices K of size $\omega(G)$ such that

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$\Gamma_G(X) \supseteq \Gamma_G(y) \supseteq K$. But we have $\Gamma_G(x) \not\supseteq K$ and $\Gamma_G(y) \not\supseteq K$. Hence we can find an induced path of length 3 between x and y contained in $\{x \cup y \cup K\}$, a contradiction.

LEMMA 2. *Let G be a perfect graph. If there exist in G two vertices x and y such that there is no induced odd path between x and y, then the graph G' obtained from G by identifying x and y is perfect.*

(This property was first proved by J. Fonlupt and J. P. Uhry [7]).

PROOF. Obviously we need only to prove that $\gamma(G') = \omega(G')$ as the property can be extended immediately to proper subgraphs of G'. Now by Lemma 1, we need only to prove that $\gamma(G) = \gamma(G')$.

Consider a coloration of G in $\omega(G)$ colours. If x and y receive the same colour, this coloration defines obviously the required coloration of G'. If x and y receive different colours, say α and β, note that y cannot belong to the connected (α, β)-component of G containing x. Otherwise we would find inside this component an induced path of odd length between x and y.

Then interchanging colours α and β on this component we find a coloration of G with $\omega(G)$ colours such that x, y have the same colour.

THEOREM 3. *Let G be a critical imperfect graph. Between any two distinct vertices of G there exists an odd induced path.*

PROOF OF THEOREM 1. Suppose not. Then there exist two vertices x and y with no odd minimal path between them. Consider $G' = (V', E')$ obtained from G by identifying x and y. We have $\gamma(G') \geqslant \gamma(G) > \omega(G)$ and by lemma 1 $\gamma(G') > \omega(G')$.

By Lemma 2 each proper subgraph of G' must be perfect. Hence G' is critical imperfect. By the Perfect Graph Theorem (Lovasz [12]), we have $\omega(G).\alpha(G) = |V| - 1$ and $\omega(G').\alpha(G') = |V'| - 1 = |V| - 2$, hence by Lemma 1 $\omega(G).(\alpha(G) - \alpha(G')) = 1$. A contradiction, as $\omega(G) \geqslant 2$.

THEOREM 4. *Let G be a graph such that every induced subgraph G' (with eventually $G' = G$) possesses the following property:*
There exist in G' (or in $\bar{G}'$) two vertices x and y without any odd induced path in G' (or in *$\bar{G}'$*) *joining x and y, then G is perfect.*

PROOF. This is an immediate consequence of Theorem 3 and of the fact that a graph is critical imperfect if and only if its complement is critical imperfect (see Lovász [12])

We call *quasi parity* graph (Q.P. for short) a graph G such that every induced subgraph possess the property described in Theorem 4, and *strictly quasi parity* graph (S.Q.P. for short) a graph such that every induced subgraph G' of G different from a complete graph possesses the following property: there exists two vertices x and y such that there is no odd induced path in G' between x and y.

It is left to the reader to check that the class of S.Q.P. graphs is strictly contained in the class of Q.P. graphs.

THEOREM 5. *Let G be a graph such that every odd cycle has two chords, then G is a strictly quasi parity graph.*

PROOF. The proof is by induction on the number of vertices of G. The theorem is trivially true if G is complete. Suppose G is not complete. Let x be a vertex of G such that $\Gamma_G(x)$ is not a complete graph, in $\Gamma_G(x)$ there exists (by virtue of the induction hypothesis) two vertices x, y such that there is no odd induced path with endpoints x and y contained in $\Gamma_G(x)$. By a variation of a lemma contained in [12], which is by the way, rather easy to demonstrate, all induced odd paths between x and y in g are (under the hypothesis of the theorem) contained in $\Gamma_G(x)$. Hence there is no odd induced path between x and y in G and G is S.Q.P.

Following V. Chvátal [6], a graph is said to be *perfectly orderable* if and only if there exists a linear order on the vertices (which induces a unique orientation for the edges of G) such that the configuration described in Figure 1 does not appear as an induced subgraph.

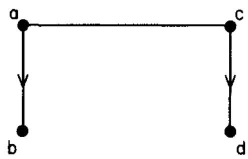

FIGURE 1.

H is a graph on four vertices with directed edges (a, b), (c, d), (a, c) or (a, b), (c, d), (c, a).

LEMMA 6. *Let G be a perfectly ordered graph. Let x, y be two vertices of G such that there is an odd induced undirected path between x and y. Then if x is the terminal vertex of the arc of the path which is incident to it, then y is the initial vertex of the arc of the path incident to y.*

PROOF. Obvious, if the undirected path is of length 3 (otherwise its orientation would give the forbidden configuration) and the remaining of the proof is easy by induction on the length of the path.

THEOREM 7. *A perfectly orderable graph is strictly quasi parity.*

PROOF. Let $G = (V, E)$ be a perfectly ordered graph. We need only to prove that, if G is different from a complete graph, there are two vertices x and y with no odd induced path between them. By induction on the number of vertices, we may suppose that the property holds for all proper induced subgraphs of G.

Suppose that there exist two sinks in G (i.e. two vertices x and y with $\Gamma_G^+(x) = \Gamma_G^+(y) = \varnothing$). By Lemma 6, there cannot exist an odd induced undirected path between x and y.

Let then x be the unique sink of G. If $\Gamma_G^-(x) = V - x$, $G_{V\setminus x}$ is a S.Q.P. graph (by induction on the number of vertices). Hence G is a S.Q.P. graph.

If $\Gamma_G^-(x) \neq V - x$, then there exists some vertex y with $y \notin \Gamma_G^-(x)$ and $\Gamma_G^+(y) \subset \Gamma_G^-(x)$. Indeed the linear order induces an acyclic orientation for G. As $|V|$ is finite, if the preceeding property did not hold, we would find another sink, by a repeated application of the following procedure: from a vertex $z' \notin \Gamma_G^-(x)$ go to any z' successor of z with $z' \notin \Gamma_G^-(x)$.

Then by Lemma 6, an odd induced path from x to y must intersect $\Gamma_G^+(y)$, contradicting $\Gamma_G^+(y) \subset \Gamma_G^-(x)$. Thus x and y are the required vertices in G.

4. FURTHER PROBLEMS AND REMARKS

(a) P. Hell [10] remarked that one can enlarge any known class of perfect graphs $\mathscr{P}$, defining a new class $\mathscr{P}'$ as follows: For any graph G belonging to $\mathscr{P}'$ and for any induced subgraph G' of G one of the two following properties holds:

(i) G' belongs to $\mathscr{P}$.
(ii) There exists in G' or in $\bar{G}'$ two vertices x, y which are not linked by an odd induced path.

(b) PROBLEMS

Problem 1 Are Q.P.-graphs or S.Q.P.-graphs recognizable in polynomial time?

Problem 2 Is it possible to find a characterization of Q.P. critical graphs (or S.Q.P. critical graphs)? (A graph is said to be Q.P. critical (resp. S.Q.P. critical) if every proper subgraph is a Q.P. graph (resp. a S.Q.P. graph) and G is not.

REMARK. According to F. Maffray [15] there exists an infinite family of perfect graphs (which are not Q.P.) We give one for instance with nine vertices x_{ij}, $1 \leqslant i \leqslant 3$, $1 \leqslant j \leqslant 3$ and edges (x_{ij}, x_{kl}) when $i \neq k$ and $j \neq l$ and according to C. Champetier [4] or C. T. Hoang and R. Hayward [9] there are infinite families of S.Q.P. critical graphs (distinct from an odd hole or an antihole of length at least 6).

Problem 3 Are strongly perfect graphs Q.P.?
A graph is said to be strongly perfect if and only if for each subgraph G' there exists an independent set which meets every maximal (for inclusion) complete subgraph in G'. (See Berge and Duchet [3]).

Problem 4 V. Chvátal [5] defined a new class of perfect graph denoted by Bip*. Is it true that all graphs in Bip* are Q.P. graphs? In particular is it true for particular subclasses of BiP* such as weakly triangulated graphs defined by R. Hayward [8] or alternation graphs defined by C. T. Hoang [11] (alternation graphs are these graphs which can be directed in such a way that on every hole the orientation of edges alternates)?

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